MOTIVES OF GRAPH HYPERSURFACES WITH TORUS OPERATIONS

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Abstract. We investigate graph hypersurfaces and study conditions under which graph hypersurfaces admit algebraic torus operations. This leads in principle to a computation of graph motives using the theorem of Bialynicki-Birula, provided one knows the fixed point loci in a resolution of singularities.

Introduction

Feynman diagrams and their amplitudes are of fundamental importance in perturbative quantum field theory. Extensive calculations of these amplitudes for graphs of low loop numbers by Broadhurst and Kreimer in [6] and [7] revealed the motivic nature of these amplitudes, showing that in many cases they are expressible as rational linear combinations of multiple zeta values. This brought up the question whether all Feynman amplitudes evaluate to multiple zeta values. By general principles [11, 16], this would mean that Feynman amplitudes are periods of mixed Tate motives. Kontsevich [15] related this to point counting on the hypersurface defined by the singularities of the integrand in the Feynman amplitude. Despite the empirical evidence created by Stembridge in [17], Belkale and Brosnan [2] showed that the point counting function for general graph hypersurfaces is not of polynomial type, in fact, it is of the most general type one can conceive in the world of motivic counting functions. Bloch, Esnault and Kreimer [5] investigated the foundations of Feynman amplitudes and their relations to periods of mixed Hodge structures, and studied the mixed Hodge structure of the middle cohomology for wheel-type graphs. Explicit graphs not of mixed Tate type have first been found by Brown-Schnetz [9, 10] and Doryn [12].

The intention of this paper is to explore torus actions on graph hypersurface X_Γ and their non-singular models, and to provide a set-up to compute the resulting motive using the theorem of Bialynicki-Birula [3]. In section §1 we study criteria for the existence of algebraic torus operations. In §2 we focus on a particular class of graphs, obtained by a glueing process, where the torus action is evident. In §3, we use the derived category $DM(k)$ of motives and apply the theorem of Bialynicki-Birula in a motivic context in order to study the motive of X_Γ. The presence of a torus action reduces the complexity of the motive of X_Γ with this method to that of the fixed point loci in some resolution of singularities.

1. Existence of torus actions on graph hypersurfaces

Definition 1.1. *Let Γ be a finite, connected, not necessarily simple graph. Then the graph polynomial $P(\Gamma)$ is defined as*

$$P(\Gamma) := \sum_T \prod_{e \notin T} X_e,$$

where T runs through all spanning trees of Γ, and X_e is a polynomial variable for each $e \in E(\Gamma)$. The polynomial $P(\Gamma)$ is homogenous of degree $h_1 = h_1(\Gamma)$ [5]. *We*

define the graph hypersurface

$$X(\Gamma) := \{P(\Gamma) = 0\} \subset \mathbb{P}^{n-1}, \quad n = \sharp E(\Gamma).$$

In [5] this polynomial was rewritten in terms of a determinant of a symmetric $(h_1 \times h_1)$-matrix $M(\Gamma)$ with linear entries. Since much of this paper relies on this description we will repeat it here. For Γ we choose an orientation of its edges. Define a map $\partial \ : \ \mathbb{Z}^{E(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$, by $e \mapsto \sum_{v \in V(\Gamma)} sgn(v, e)v$, where $sgn(v, e) = 1$ if v is the source of the edge e, further $sgn(v, e) = -1$ if v is the target of E. This gives rise to a simplicial complex $\mathbb{Z}^{E(\Gamma)} \xrightarrow{\partial} \mathbb{Z}^{V(\Gamma)}$ and a corresponding exact sequence

$$0 \to H_1(\Gamma, \mathbb{Z}) \xrightarrow{\iota} \mathbb{Z}^{E(\Gamma)} \to \mathbb{Z}^{V(\Gamma)} \to H_0(\Gamma, \mathbb{Z}) \to 0.$$

Let $l_e(\cdot)$, $e \in E(\Gamma)$ denote the dual basis of $e \in E(\Gamma) \subseteq \mathbb{Z}^{E(\Gamma)}$. Then we can consider the bilinear forms b_e of rank 1 given by

$$b_e := (l_e \circ \iota) \cdot (l_e \circ \iota) \ : \ H_1(\Gamma, \mathbb{Z}) \times H_1(\Gamma, \mathbb{Z}) \to \mathbb{Z}.$$

Denote by M_e the symmetric matrix associated to b_e and set

$$M(\Gamma) := \sum_{e \in E(\Gamma)} X_e M_e \in \mathbb{Z}[X_e : e \in E(\Gamma)]_1 \otimes_{\mathbb{Z}} \mathrm{End}(\mathbb{Z}^{h_1(\Gamma)}).$$

Here $\mathbb{Z}[X_e : e \in E(\Gamma)]_1$ denotes the degree 1 part of the algebra $\mathbb{Z}[X_e : e \in E(\Gamma)]$.

Lemma 1.2. *One has* $P(\Gamma) = \det M(\Gamma)$.

Proof. See [[5], Proposition (2.2)]. $\square$

Remark 1.3. $P(\Gamma)$ *satisfies the deletion-contraction formula*

$$P(\Gamma) = X_e P(\Gamma \backslash e) + P(\Gamma/e),$$

where $e \in E(\Gamma)$ *and* Γ/e *is the quotient of graphs. From this identity one can immediately deduce that subdivision of edges gives rise to affine fibre bundles over* X_Γ*: Let* Γ' *be the graph obtained from* Γ *by subdividing the edge* e *into* e_1 *and* e_2*. Then*

$$P(\Gamma') = P(\Gamma)(X_{e_1} + X_{e_2}).$$

Form now on we choose a field k of characteristic zero and consider $M(\Gamma)$ as a matrix in $k[X_e : e \in E(\Gamma)]_1$.

Lemma 1.4. *Let* Γ *be a graph and* $M(\Gamma)$ *as above. Then the diagonal entries of* $M(\Gamma)$ *are linearly independent in* $k[X_e : e \in E(\Gamma)]_1$.

Proof. It is well known that the edges in the complement of a spanning tree are in 1-1 correspondence with loop subgraphs which form a basis of $H_1(\Gamma, \mathbb{Z})$. To see this, note that adding an edge one obtains exactly one cycle in $H_1(\Gamma)$. This implies the existence of a loop which contains an edge e not occuring in any other loop subgraph of this spanning tree T. The next edge we choose to be in the set $\Gamma \backslash (T \cup \{e\})$. Iterating this process one ends up with an edge which does not occur in any other cycle but the last one to be constructed in this way. Hence there exists at least one cycle with external edge. From this the assertion follows by an easy induction. $\square$

Definition 1.5 (Weight lattice)**.** *Let* k *be a field,* $s \in \mathbb{N}_0$ *and* $k[X_1, \ldots, X_n]_s$ *be the vector space of homogeneous polynomials in* m *variables of degree* s*. For any*

$$f = \sum_{|\alpha| = s} c_\alpha X^\alpha \in k[X_1, \ldots, X_n]_s$$

let

$$\Gamma(f) := \{\eta \in \mathbb{Z}^n \ : \ \eta \cdot \alpha = 0 \ \textit{for any} \ \alpha \ \textit{with} \ c_\alpha \neq 0\}$$

be the weight lattice of the polynomial f. Here, by a lattice we mean a free $\mathbb{Z}$-module of finite rank.

Definition 1.6 (Torus operation)**.** *We define a (faithful) torus operation on a subvariety $X \subset \mathbb{P}^{n-1}$ to be a (mono)morphism*

$$\mathbb{G}_m^r \longrightarrow \mathrm{Aut}(X) \cap \mathrm{PGL}_n(k)$$

from a split r-dimensional torus $\mathbb{G}_m^r$ into the linear automorphisms of X.

From now on, we assume that $X = \{f = 0\} \subset \mathbb{P}^{n-1}$ is a hypersurface. If k is algebraically closed, one can always choose a coordinate system $Z_1, ..., Z_n$ where $\mathbb{G}_m^r$ acts diagonally and the torus operation is encoded into the weight lattice $\Gamma(f)$ of f in this coordinate system. The number $\mathrm{rank}(\Gamma(f)) - 1$ is the maximal rank r of a diagonalized torus operation.
We will be mainly interested in the case where the polynomial f is the determinant of a symmetric $(h \times h)$-matrix M with linear polynomials as entries. In that case let $m = \binom{h+1}{2}$ and think of the elements in $\mathbb{Z}^m$ as upper triangle entries of M and write $\omega = (\omega_{ij})_{1 \le i,\, j \le h} \in \mathbb{Z}^m$ with $\omega_{ij} = \omega_{ji}$. In this case, we define the lattice

$$\Lambda_h := \{\omega = (\omega_{ij})_{1 \le i,\, j \le h} \in \mathbb{Z}^m \ : \ 2\omega_{ij} = \omega_{ii} + \omega_{jj}\}.$$

The diagonal sublattice $\Delta \hookrightarrow \Lambda_h$, where all entries are equal, is irrelevant since it corresponds to the action of the center of the $\mathrm{SL}(n,\, k)$. The lattice Λ_h has rank h, since an element is already determined by the diagonal entries. It is a weight lattice of a certain polynomial for the following prototype example:

Example 1.7. *Suppose $M = (Y_{ij}) \in k[X_1, \dots, X_n]^{h \times h}$ is a symmetric matrix such that all entries in the upper triangle are non-zero linearly independent linear homogenous polynomials. Then consider the maximal torus*

$$\mathbb{G}_m^{h-1} \simeq \{\mathrm{diag}(t_1, \dots, t_h) \in k^{h \times h} \ : \ \prod_{i=1}^{h} t_i = 1\} \subseteq \mathrm{SL}_h(k)$$

given by diagonal matrices. Then the variety

$$X := \{\det(M) = 0\} \subseteq \mathbb{P}^{n-1}$$

carries a faithful $\mathbb{G}_m^{h-1}$-action, defined by $Y_{ij} \mapsto t_i t_j Y_{ij}$, since one has

$$\det(\mathrm{diag}(t_1, \dots, t_h) M \mathrm{diag}(t_1, \dots, t_h)) = \det(M) \prod_{i=1}^{h} t_i^2 = \det(M).$$

The weight lattice of this polynomial is obviously given by the lattice Λ_h.

The following is a version of the previous example with much weaker assumptions:

Theorem 1.8. *Let k be a field. Let $M \in k[X_1, \dots, X_n]^{h \times h}$ be a symmetric matrix such that there are $l(M)$ non-zero entries in the upper triangle which are linearly independent linear homogenous polynomials, and all diagonal entries are non-zero. Then the hypersurface*

$$X := \{\det(M) = 0\} \subset \mathbb{P}^{n-1}(k),$$

admits a linear, faithful $\mathbb{G}_m(k)^r$-action with $r \geq h - 1 + n - l(M)$.

Remark 1.9. *The number $r = h - 1 + n - l(M)$ is maximal with this property for the torus action defined in Example 1.7, but there may be additional actions. We construct a torus operation in such way that the weight lattice is contained in Λ_h.*

Proof. We proceed via induction on h and we prove that the weight lattice of M is contained in Λ_h. It suffices to construct a torus operation in the set of variables Y_{ij}, which we identify with the non-zero matrix entries of M by abuse of notation. This amounts to a linear change of variables only.

In case $h = 1$, there is nothing to prove. In case $h = 2$, then $M = \begin{pmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{pmatrix}$ or $M = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{22} \end{pmatrix}$. Then $\omega := (0, 2, 1)$ defines a non-zero weight vector of a diagonally acting $\mathbb{G}_m$ on the entries in the variables Y_{11}, Y_{22}, Y_{12}. Note that this weight vector satisfies the relations $2\omega_{ij} = \omega_{ii} + \omega_{jj}$ for all $i, j \in \{1, 2\}$, i.e., it is contained in Λ_2.

Case $h > 2$: Let M' be the matrix obtained from M be deleting the first line and column. The matrix satisfies the assumption of Thm. 1.8. By induction, let $\Omega' \in \Lambda_{h-1}$ be a weight vector corresponding to the faithful torus action of rank $h - 2$ on $X' = \{\det(M') = 0\}$. Then using the equations $\omega_{1i} = \frac{\omega_{11}+\omega_{ii}}{2}$ we fill up Ω' to a weight matrix

$$\Omega = \begin{pmatrix} \omega_{11} & \frac{\omega_{11}+\omega_{22}}{2} & \cdots & \frac{\omega_{11}+\omega_{hh}}{2} \\ \frac{\omega_{11}+\omega_{22}}{2} & & & \\ \vdots & & \Omega' & \\ \frac{\omega_{11}+\omega_{hh}}{2} & & & \end{pmatrix},$$

where $\omega_{11} \in \omega_{22} + 2\mathbb{Z}$ may be chosen arbitrarily. We know by Example 1.7 that the determinantal hypersurface X is certainly invariant under the action defined by Ω, since $\Omega \in \Lambda_h$.
The assumption on the diagonal guarantees that the first row of M contains a non-zero entry M_{11}, hence the action defined by Ω is faithful on the projective subspace defined by the entries in the upper triangle of M. $\square$

Example 1.10. *Wheels WS_h with h spokes and $2h$ edges satisfy the Proposition, since*

$$M(WS_h) = \begin{pmatrix} Y_{11} & -X_2 & 0 & \cdots & -X_1 \\ -X_2 & Y_{22} & -X_3 & \cdots & 0 \\ 0 & -X_3 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & & Y_{h-1,h-1} & -X_h \\ -X_1 & 0 & \cdots & -X_h & Y_{hh} \end{pmatrix},$$

with $Y_{ii} = X_i + X_{i+1} + X_{h+i}$. Here $i + 1$ is to be considered $\mod h$.
As a consequence, the associated hypersurfaces X_h carries faithful torus operation of rank $h-1$. This bound is sharp, e.g., in the case $h = 3$, the hypersurface $X_3 \subseteq \mathbb{P}^5$ is the complement of the 5-dimensional homogenous space $PSL_3(\mathbb{C})/SO_3(\mathbb{C})$, which carries a faithful rank 2 torus operation, since the group $PSL_3(\mathbb{C})$ has rank 2 [4].

In general, the condition of linear independence in Thm. 1.8 is too restrictive. We need to define a new invariant for symmetric matrices M to formulate a more general result. The proof of Thm. 1.8 then implies much more as we will see now. Let k be a field. Let $M \in k[X_1, \dots, X_n]_1^{h\times h}$ be a symmetric matrix of linear forms such that all diagonal entries are non-zero and linearly independent. We denote by $l(M)$ the dimension of the span of all upper-triangular entries, and by N the number of all non-zero upper-triangular entries. By a linear change of variables, we may assume that each non-zero entry M_{ij} of M is either a variable $X_1, ..., X_{l(M)}$ or

a linear form $L_{ij}(X_1, ..., X_{l(M)})$ in those variables. All diagonal entries are therefore assumed to be independent variables.

Definition 1.11. *We define an equivalence relation on indices (ij) $(i \le j)$ of the non-zero entries M_{ij} as the transitive hull of the symmetric relation given by*

$$(ij) \sim (kl) \Leftrightarrow \text{ there is a variable } X \in \{X_1, ..., X_n\} \text{ in } M_{ij} \text{ and } M_{kl}.$$

The equivalence classes are called clusters.
An element (ij) in a cluster C is called excessive, if X_i or X_j do not occur in $L_{ij}(X_1, ..., X_{l(M)})$. Let

$$\delta(M) := \sum_{\text{clusters } C} (|C| - 1) + \sharp \text{ excessive entries in } M$$

be the excess of M.

Theorem 1.12. *Under these assumptions, the dimension r of a linear, faithful $\mathbb{G}_m(k)^r$-action on the hypersurface*

$$X := \{\det(M) = 0\} \subset \mathbb{P}^{n-1}(k),$$

which is diagonal in the variables $X_1, ..., X_n$ is at least

$$r = \max\left(0, h - 1 + n - \ell(M) - \delta(M)\right).$$

Proof. Substituting new variables for each linear form $L(X_1, ..., X_{l(M)})$, we arrive at an inclusion

$$i : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{N+n-\ell(M)-1},$$

where $N - \ell(M)$ is the number of additional variables Y_{ij} with $i > j$. This inclusion maps X to a codim $N - \ell(M) + 1$ subvariety $X' = i(X) = \{\det(M') = 0\} \bigcap \{H_{ij} = 0\}$, where M' is the matrix obtained by the same substitutions, and H_{ij} are the linear hyperplanes

$$H_{ij} : Y_{ij} = L_{ij}(X_1, ..., X_{l(M)}).$$

Theorem 1.8 implies the existence of a torus T of rank $\ge h - 1 + N + n - \ell(M) - \ell(M') = h - 1 + n - l(M)$ acting on $\{\det(M') = 0\}$. Now we count conditions to obtain the codimension of a torus stabilizing $X' = i(X)$. For the variables X_i in each cluster C to have equal weight amounts to $|C| - 1$ conditions. The weights of the new variables Y_{ij} are determined by diagonal entries, hence give no new condition on the torus operation if L_{ij} is not excessive.
In total, this gives $\delta(M)$ conditions, and hence we obtain a torus operation of rank $\ge n - \ell(M) + h - 1 - \delta(M)$. □

Remark 1.13. *The bounds in this theorem are far from being sharp. We provide a counter example below. In general, the proof give an algorithm to compute the correct rank of a torus, by checking whether the weight of the entry Y_{ij} coincides with the weight of the variables occuring in L_{ij}.*

In other words, in order to find a torus operation of maximal rank, we need to find a coordinate system, where the sum over cluster lengths minus one is minimized.

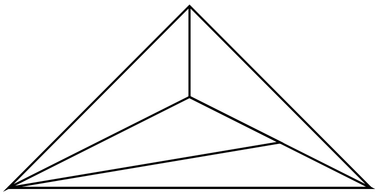
Example 1.14. *Consider the graph which is the wheel with 3 spokes with one additional triangle subdivided. This gives rise to the matrix*

$$M = \begin{pmatrix} X_2 + X_6 + X_8 & X_2 + X_6 & -X_2 & X_2 \\ X_2 + X_6 & X_1 + X_2 + X_4 + X_6 + X_7 & -X_1 - X_2 - X_4 & X_1 + X_2 \\ -X_2 & -X_1 - X_2 - X_4 & X_1 + X_2 + X_4 + X_5 & -X_1 - X_2 \\ X_2 & X_1 + X_2 & -X_1 - X_2 & X_1 + X_2 + X_3 \end{pmatrix}$$

Substituting as in Theorem 1.12 we arrive at

$$M = \begin{pmatrix} Y_1 & Y_5 & Y_8 & -Y_8 \\ Y_5 & Y_2 & Y_6 & -Y_7 \\ Y_8 & Y_6 & Y_3 & Y_7 \\ -Y_8 & -Y_7 & Y_7 & Y_4 \end{pmatrix}.$$

Obviously we have two clusters of length 2 *and* 6 *clusters of length* 1*. By the theorem this means we can expect* $X_\Gamma = \{\det(M) = 0\} \subseteq \mathbb{P}^7$ *to have no torus action. However, there is a* 1*-dimensional action given by the weight vector* $\omega := (3, -1, -1, -1, 1, -1, -1, 1)$*. The algorithms would give the same result, as* Y_7 *and* Y_8 *are in excessive positions but impose no extra relation.*



2. Examples: $*$-graphs

A class of examples which have linearly independent entries in $M(\Gamma)$ and which contains the wheels with n spokes are the $*$-graphs:

Definition 2.1. *A polygonal graph* Γ *is a connected, not necessarily simple, graph which has a decomposition* $\Gamma = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_h$ *as a sucessive gluing along non-empty, connected sets of edges inside the polygons* Δ_i*, and such that no edge is used twice for glueing. Let* E_0 *be the union of all edges used for the glueing. A* $*$*-graph* Γ *is a polygonal graph such that every such decomposition has the property* $h_1(E_0) = 0$.

Example 2.2. *The example of a banana graph with* 4 *leaves shows that the condition on* E_0 *to be simply-connected depends on the glueing order. The matrix* $M(\Gamma)$ *has linearly dependent entries for this graph. This shows that we have to require some strong conditions on the glueings.*

Lemma 2.3. *For any* $*$*-graph* Γ *one has*

$$h_1(\Gamma) = \sharp \text{ polygons } \Delta_i = h.$$

Proof. We use the Mayer-Vietoris Theorem and induction on the number of polygons. Assume $\Gamma = \Gamma' \cup \Delta$, where Δ is a polygon. Then the intersection $\Gamma' \cap \Delta$ is a connected and contractible union of edges, in particular $h_1(\Gamma' \cap \Delta) = 0$ and $h_0(\Gamma' \cap \Delta) = 1$. Hence there is an isomorphism $H_1(\Gamma') \oplus H_1(\Delta) \cong H_1(\Gamma)$. □

Proposition 2.4. *Let* Γ *be a polygonal graph. Then the following conditions are equivalent:*
(i) Γ *is a* $*$*-graph.*
(ii) The non-zero upper-triangular matrix entries M_{ij} *of* $M(\Gamma)$ *are linearly independent polynomials in* $k[X_1, ..., X_n]_1$.

The statement in (ii) is independent of the choice of a basis of $H_1(\Gamma, \mathbb{Z})$.

Proof. (ii) $\Rightarrow$ (i): Assume $\Gamma = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_h$, but $h_1(E_0) > 0$. Let $\Delta_1, ..., \Delta_h$ be the natural basis of $H_1(\Gamma)$ given by the polygons Δ_i. Given a simple non-zero loop $\gamma \subset E_0$, there is a linear relation between the diagonal entries for all Δ_i meeting γ and all off-diagonal entries carrying glueing data for these Δ_i.
(i) $\Rightarrow$ (ii): Conversely, suppose that Γ is a $*$-graph and we have given a linear

relation among the entries of $M(\Gamma)$. By definition of $*$-graphs this relation involves a diagonal element, since every edge is only used once for glueing. Hence, we get an equations

$$\sum_{i=1}^{h} a_i M_{ii} = \sum_{i<j} b_{ij} M_{ij},$$

with at least one a_i and one b_{ij} non-zero by Lemma 1.4. This is a contradiction, since each Δ_i occurring on the left with $a_i \neq 0$ has an edge which is not contained in E_0. □

Corollary 2.5. *Any $*$-graph admits a faithful (h_1-1)-dimensional torus operation.*

Proof. By Prop. 2.4, the entries of $M(\Gamma)$ satisfy the assumptions of Prop. 1.8. □

3. Motivic Bialynicki-Birula decompositions

In this section we discuss how to apply high dimensional torus actions on X_Γ to compute the motive of a graph hypersurface $X_\Gamma = \{\det M(\Gamma) = 0\}$ in specific examples.

The simplest example which is not entirely trivial is $\Gamma = WS_3$, the wheel with 3 spokes. The graph hypersurface X_Γ for $\Gamma = WS_3$ is isomorphic to $\mathrm{Sym}^2\mathbb{P}^2$, which has a resolution by blowing up the diagonal and admits a 2-dimensional torus operation. The motive of X_Γ is mixed Tate by [4, Sect. 9]. We want to generalize this to a larger class of graphs.

Lemma 3.1. *Let Γ be a graph such that the non-zero entries in the upper triangular part of the matrix $M(\Gamma)$ are linearly independent. Then for the action of $T := \mathbb{G}_m^r$ with $r = h_1(\Gamma)-1+n-\ell(M)$ on X_Γ, as described in Theorem 1.8 and Example 1.7, the variety $\mathrm{Fix}_{\mathbb{P}^{\sharp E(\Gamma)-1}}(T)$ consists of points contained in X_Γ.*

Proof. We may assume that $n = \ell(M)$, since the action on the $n - \ell(M)$ extra variables is effective. By Example 1.7 the action on the generic symmetric matrix with independent linear entries is given by $(t, x) \mapsto (t_i t_j x_{ij})$. Choosing special values for t_i and t_j with $\prod_i t_i = 1$, one sees that the fixed points in this case are just the points corresponding to the usual standard basis of the underlying space $\mathbb{P}^{N-1}$ with $N = \binom{h_1-1}{2}$. In general, the graph hypersurfaces of the type described in the assumption are intersections of the generic zero set of the determinant of this generic symmetric matrix with (T-invariant) linear coordinate subspaces. The induced action is faithful (e.g. by Lemma 1.4). Hence the fixed point set $\mathrm{Fix}_{\mathbb{P}^{\sharp E(\Gamma)-1}}(T)$ is given by points in $\mathbb{P}^{N-1}$ with exactly one non-zero entry supported in $E(\Gamma)$. Obviously these points are contained in X_Γ. □

Note that all graph hypersurfaces of wheels WS_n with n spokes satisfy this lemma. In the following we use motives $M(X)$ in the sense of Voevodsky's triangulated category $\mathrm{DM}(k) = \mathrm{DM}_{gm}(k)$ [18] for any k-scheme X.

Definition 3.2. *Let $f : Z \to X$ be a closed immersion of schemes defined over k. Then the relative motive corresponding to this morphism is the mapping cone $M(X, Z)$ extending the diagram $M(Z) \to M(X)$ to a distinguished triangle in $\mathrm{DM}(k)$:*

$$M(Z) \to M(X) \to M(X, Z).$$

Now we want to give a criterion when a graph motive $M(X_\Gamma) \in \mathrm{DM}(k)$ is mixed Tate. In view of the classical Bialynicki-Birula theorem [3] and its motivic versions [8], one might expect that the motive should be determined by the fixed point loci. In the presence of singularities, the stratification via affine bundles in the Bialynicki-Birula proof becomes degenerate, so that one cannot apply the same

idea. However, we can describe $M(X_\Gamma)$ using "smaller" motives in a resolution of singularities. In order to do this, let X be an arbitrary projective k-variety with a faithful $T := \mathbb{G}_m$-action, and $\pi = \pi_1 \circ \ldots \circ \pi_{m-1}$ be a resolution of singularities together with a stratification

$$\emptyset \subset X_m \subset \ldots \subset X_1 \subset X_0 = X$$

such that

(1) $X_i \backslash X_{i+1}$ is smooth for all $1 \leq i \leq m-1$,
(2) $\pi_i \circ \ldots \circ \pi_{m-1} \colon X^{(i)} \to X$ (here $X^{(i)}$ denotes the abstract blow-up of X given by $\pi_i \circ \ldots \circ \pi_{m-1}$) resolves the singularities up to X_i, i.e., $(\pi_i \circ \ldots \circ \pi_{m-1})^{-1}(X_i) \subseteq X^{(i)}$ is a union of smooth components with smooth mutual intersections, and $\pi_i \circ \ldots \circ \pi_{m-1}$ is an isomorphism outside $(\pi_i \circ \ldots \circ \pi_{m-1})^{-1}(X_{i+1})$.
(3) Each π_j is equivariant with respect to the T-action.

Such a situation can always be obtained in case k is an algebraically closed field with $\mathrm{char}(k) = 0$ by using embedded resolution of singularities [19]. We assume that we are in this situtation now.

Proposition 3.3. *Let*

$$\tau := \langle M(F),\, M(F^{(i)}) \colon \forall i,\, F \subseteq \mathrm{Fix}_X(T), F^{(i)} \subseteq \mathrm{Fix}_{X^{(i)}}(T) \textit{ irred. components}\rangle$$

be the full triangulated subcategory of $\mathrm{DM}(k)$ *obtained by taking the pseudoabelian envelop inside* $\mathrm{DM}(k)$ *of the full triangulated subcategory, closed under Tate twists, generated by the motives in the brackets. Then* $M(X) \in \tau$. *In particular if the generating motives lie in* $\mathrm{DTM}(k)$ *then so does* $M(X)$.

Proof. Note that for any triangulated category τ and any sequence $A_r \to A_{r-1} \cdots \to A_1 \to A_0$ of arrows in τ, A_0 is contained in the full triangulated subcategory τ_0 of τ generated by A_r and $\mathrm{cone}(A_i \to A_{i-1})$, for $1 \leq i \leq r$. We apply this fact to the sequence $M(X_r) \to M(X_{r-1}) \to \cdots \to M(X)$.

Let Y be any k-scheme. Let $\pi \ : \ \widehat{Y} \to Y$ be an abstract blow-up with center $Z \hookrightarrow Y$ in the sense of [18, Prop 4.1.3]. In this case there is a distinguished triangle

$$M(\pi^{-1}(Z)) \to M(\widehat{Y}) \oplus M(Z) \to M(Y)$$

in $\mathrm{DM}(k)$. The diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & M(Z) \arrow[r, "id"] \arrow[d] & M(Z) \arrow[d] \\
M(\pi^{-1}(Z)) \arrow[r] \arrow[d] & M(\widehat{Y}) \oplus M(Z) \arrow[r] \arrow[d] & M(Y) \arrow[d] \\
M(\pi^{-1}(Z)) \arrow[r] & M(\widehat{Y}) \arrow[r] & M(Y,Z)
\end{tikzcd}$$

implies, using Verdier's lemma [1, Prop. 1.1.11], the Aeppli formula

$$M(\widehat{Y}, \pi^{-1}(Z)) \cong M(Y,Z)$$

in $\mathrm{DM}(k)$.

Let $A \hookrightarrow W$ be a closed immersion of smooth projective varieties defined over k. Let $T = \mathbb{G}_m$ be an algebraic torus acting faithfully on W such that $A \hookrightarrow W$ is equivariant. Then

$$M(W,\, A) \simeq \bigoplus_F M(F^+,\, F^+ \cap A),$$

where F runs through all components of the fix point locus such that the corresponding cell F^+ in the sense of [13] is not contained in A. The proof follows from the commutativity of the diagram

$$\begin{tikzcd}
M(A) \arrow[r, "\cong"] \arrow[d] & \bigoplus_F M(F^+ \cap A) \arrow[d] \\
M(W) \arrow[r, "\cong"] \arrow[d] & \bigoplus_F M(F^+) \arrow[d] \\
M(W, A) \arrow[r] & \bigoplus_F M(F^+, F^+ \cap A),
\end{tikzcd}$$

where the first two isomorphisms follow from the motivic version of Bialynicki-Birula [8, 13]. This statement generalizes to the case where W and A form a relative normal crossing variety. It suffices to check this for the non-relative version of Bialynicki-Birula. Write $W = W_1 \cup W_2$, where W_1 is irreducible. We want to show $M(W) = \bigoplus_F M(F^+)$. We get the diagram:

$$\begin{tikzcd}
M(W_1 \cap W_2) \arrow[r] \arrow[d, "\cong"] & M(W_1) \oplus M(W_2) \arrow[r] \arrow[d, "\cong"] & M(W) \arrow[d, dashed, "\cong"] \\
\bigoplus_F M(F^+ \cap W_1 \cap W_2) \arrow[r] & \bigoplus_F M(F^+ \cap W_1) \oplus \bigoplus_F M(F^+ \cap W_2) \arrow[r] & \bigoplus_F M(F^+)
\end{tikzcd}$$

Here the first isomorphism follows from the smoothness of the intersections of the components of W_2 with W_1 and induction and the second isomorphism follows from the usual Bialynicki-Birula and induction. By the functoriality of the Bialynicki-Birula decomposition the second row is just a direct sum of distinguished Mayer-Vietoris triangles. This implies

$$M(W) = \bigoplus_F M(F^+).$$

Now the assertion of the proposition follows from successively resolving equivariantly the singularities of X and each X_i.

$$\begin{aligned}
M(X_i, X_{i+1}) &\cong M(X_i^{(i)}, X_{i+1}^{(i)}) \\
&\cong \bigoplus_F M((F^{(i)})^+ \cap X_i^{(i)}, (F^{(i)})^+ \cap X_{i+1}^{(i)})(n_\bullet)
\end{aligned}$$

by the Aeppli formula and the relative Bialynicki-Birula decomposition.
Further $M((F^{(i)})^+ \cap X_i^{(i)}, (F^{(i)})^+ \cap X_{i+1}^{(i)})$ is the mapping cone of

$$M((F^{(i)})^+ \cap X_{i+1}^{(i)}) \to M((F^{(i)})^+ \cap X_i^{(i)}).$$

Finally $M((F^{(i)})^+ \cap X_{i+1}^{(i)}) \cong M(F^{(i)} \cap X_{i+1}^{(i)})(n_\bullet)$ and $M((F^{(i)})^+ \cap X_i^{(i)}) \cong M(F^{(i)} \cap X^{(i)})(n_\bullet)$, where $n_\bullet$ indicates an appropriate Tate twist.

□

Remark 3.4. *To be more precise, an object $M \in \mathrm{DM}(k)$ is called mixed Tate, if it is in the image of*

$$\mathrm{DTM}(k) \to \mathrm{DM}(k) \otimes \mathbb{Q},$$

where $\mathrm{DTM}(k)$ is the $\mathbb{Q}$-linear triangulated category defined by Levine [16].

Poropsition 3.3 reduces the complexity of the motive of X_Γ with this method to that of the fixed point loci in some resolution of singularities. This method should be successful provided there is some sufficiently high dimensional torus action. However, besides the wheel with 3 spokes, we do not have much evidence yet, as the computational complexity is quite large even in simple examples.

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